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van den Berg, M., & Carroll, T. (2020). On the torsion function with mixed boundary conditions. *Potential Analysis*, *55*, 277-284. https://doi.org/10.1007/s11118-020-09857-1

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Potential Analysis
https://doi.org/10.1007/s11118-020-09857-1

On the Torsion Function with Mixed Boundary Conditions

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Received: 5 November 2019 / Accepted: 31 May 2020 / Published online: 16 June 2020
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Abstract
Let D be a non-empty open subset of $\mathbb{R}^m,\ m \geq 2$, with boundary ∂D, with finite Lebesgue measure $|D|$, and which satisfies a parabolic Harnack principle. Let K be a compact, non-polar subset of D. We obtain the leading asymptotic behaviour as $\varepsilon \downarrow 0$ of the L^∞ norm of the torsion function with a Neumann boundary condition on ∂D, and a Dirichlet boundary condition on $\partial(\varepsilon K)$, in terms of the first eigenvalue of the Laplacian with corresponding boundary conditions. These estimates quantify those of Burdzy, Chen and Marshall who showed that $D \setminus K$ is a non-trap domain.

Keywords Torsion function · Dirichlet boundary condition · Neumann boundary condition

Mathematics Subject Classification (2010) 35J25 · 35J05 · 35P15

1 Introduction and Main Results

Let D be an open, non-empty set in $\mathbb{R}^m,\ m \geq 2$, with finite Lebesgue measure $|D|$, and let $K \subset D$ be a compact set with boundary ∂K, and with positive logarithmic capacity if $m = 2$ or with positive Newtonian capacity $\mathrm{cap}\,(K)$ if $m \geq 3$. Let $u_{K,D}$ be the solution of

$$-\Delta u = 1,$$

with Dirichlet boundary condition

$$u(x) = 0,\ x \in \partial K, \tag{1}$$

and Neumann boundary condition

$$\frac{\partial u}{\partial \nu}(x) = 0,\ x \in \partial D, \tag{2}$$

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where ν is the inward normal. Boundary conditions Eqs. 1 and 2 have to be understood in the weak sense. In particular Eq. 1 holds for all regular points of ∂K. Let $\pi_D(x, y; t)$, $x \in D$, $y \in D$, $t > 0$ denote the Neumann heat kernel for D. We say that the parabolic Harnack principle (PHP for short) holds in D if for some $t_0 \in (0, \infty)$ there exists $c_0 = c_0(D, t_0) < \infty$, such that

$$\pi_D(x, y; t) \le c_0 \pi_D(v, w; t),\ t \ge t_0,\ \ x, y, v, w \in D.$$

See also [8]. As was pointed out in [4], PHP is equivalent to the following assertion: there exist $t_1 \in (0, \infty)$, $c_1 < \infty$, $c_2 > 0$ depending on D such that

$$\sup_{x,y\in D} \left| \pi_D(x, y; t) - \frac{1}{|D|} \right| \le c_1 e^{-c_2 t},\ t \ge t_1. \tag{3}$$

It was shown in [4] that if D satisfies PHP then $u_{K,D}$ is bounded, and $D \setminus K$ is a *non-trap* domain. In Theorem 1 below we quantify this statement in terms of the first eigenvalue $\lambda(K, D)$ of the Laplacian with boundary conditions Eqs. 1 and 2 in the case where K is scaled down by a factor ε with respect to a fixed point (the origin) in D.

Estimates of this type are well known for the torsion function u_Ω for an open set Ω satisfying a 0 Dirichlet boundary condition on $\partial\Omega$. In [2] it was shown that $u_\Omega \in L^\infty(\Omega)$ if and only if $\lambda(\Omega) > 0$. If the latter holds then

$$\lambda(\Omega)^{-1} \le \|u_\Omega\|_\infty \le \mathfrak{c}_m \lambda(\Omega)^{-1},$$

where $\mathfrak{c}_m$ is the sharp constant defined by

$$\mathfrak{c}_m = \sup\{\lambda(\Omega)\|u_\Omega\|_\infty :\ \Omega \text{ open in } \mathbb{R}^m, \lambda(\Omega) > 0\},$$

and $\|\cdot\|_p$ denotes the standard L^p norm, $1 \le p \le \infty$.

In [2] it was shown that $\mathfrak{c}_m \le 4 + 3m\log 2$. This bound has been improved since. See for example [5] and [10]. For general open, non-empty, and connected D, and a non-empty compact subset $K \subset D$ one does not have boundedness of $u_{K,D}$. Examples of these *trap* domains were given in [4].

Theorem 1 *Let $D \subset \mathbb{R}^m$, $m \ge 2$, be open, non-empty, containing the origin, and let D satisfy the parabolic Harnack principle. If K is a non-polar compact subset of D, then for $\varepsilon \downarrow 0$,*

$$\lambda(\varepsilon K, D)\|u_{\varepsilon K,D}\|_\infty = \begin{cases} 1 + O\big((\log \varepsilon^{-1})^{-1/2}\big), & m = 2, \\ 1 + O\big(\varepsilon^{(m-2)/2}\big), & m \ge 3, \end{cases} \tag{4}$$

where $\varepsilon K = \{y \in \mathbb{R}^m : \varepsilon^{-1} y \in K\}$. Furthermore for any non-polar compact set $K \subset D$,

$$\|u_{K,D}\|_\infty \ge \frac{1}{\lambda(K, D)}. \tag{5}$$

It was shown in Theorem 2.5(i) in [4] that if Eq. 3 holds, then the Neumann Laplacian on D has discrete spectrum. Sufficient geometric conditions for D to satisfy the PHP were obtained in, for example, Corollary 2.7 of [4]. Conversely PHP implies some geometric and spectral properties of D. The proposition below is of independent interest.

Proposition 2 *Let D be open, non-empty, with $|D| < \infty$. If* Eq. 3 *holds then we have the following.*

(i) *D is connected.*
(ii) *The first eigenvalue of the Neumann Laplacian acting in $L^2(D)$ has multiplicity 1.*

(iii)

$$\mu(B)\left(\frac{|B|}{|D|}\right)^{2/m} \geq \mu(D) \geq c_2, \tag{6}$$

where $\mu(D)$ is the first non-zero eigenvalue of the Neumann Laplacian acting in $L^2(D)$, and B is a ball of radius 1 in $\mathbb{R}^m$.

2 Proof of Theorem 1

In this section we prove Theorem 1.

Proof Let $\pi_{K,D}(x, y; t)$, $x \in D \setminus K$, $y \in D \setminus K$, $t > 0$ denote the heat kernel with a Neumann boundary condition on ∂D, and with a 0 Dirichlet boundary condition on ∂K. We have for $\delta \in (0, 1)$,

$$\begin{aligned} u_{K,D}(x) &= \int_0^\infty dt \int_{D\setminus K} dy\, \pi_{K,D}(x, y; t) \\ &= \int_0^{t_1/(1-\delta)} dt \int_{D\setminus K} dy\, \pi_{K,D}(x, y; t) + \int_{t_1/(1-\delta)}^\infty dt \int_{D\setminus K} dy\, \pi_{K,D}(x, y; t) \\ &\leq \int_0^{t_1/(1-\delta)} dt \int_{D\setminus K} dy\, \pi_D(x, y; t) + \int_{t_1/(1-\delta)}^\infty dt \int_{D\setminus K} dy\, \pi_{K,D}(x, y; t) \\ &\leq \frac{t_1}{1-\delta} + \int_{t_1/(1-\delta)}^\infty dt \int_{D\setminus K} dy\, \pi_{K,D}(x, y; t). \end{aligned} \tag{7}$$

By the heat semigroup property, and by Cauchy-Schwarz's inequality,

$$\begin{aligned} \pi_{K,D}(x, y; t) &= \int_{D\setminus K} \pi_{K,D}(x, z; t/2)\, \pi_{K,D}(z, y; t/2)\, dz \\ &\leq \left(\int_{D\setminus K} \pi_{K,D}(x, z; t/2)^2\, dz\right)^{1/2} \left(\int_{D\setminus K} \pi_{K,D}(z, y; t/2)^2\, dz\right)^{1/2} \\ &= \big(\pi_{K,D}(x, x; t)\, \pi_{K,D}(y, y; t)\big)^{1/2}. \end{aligned} \tag{8}$$

By the spectral theorem we have

$$\pi_{K,D}(x, x; t) \leq e^{-\delta t \lambda(K,D)} \pi_{K,D}(x, x; (1-\delta)t). \tag{9}$$

By Eqs. 8 and 9,

$$\begin{aligned} \big(\pi_{K,D}(x, y; t)\big)^\delta &\leq e^{-\delta^2 t \lambda(K,D)} \big(\pi_{K,D}(x, x; (1-\delta)t)\pi_{K,D}(y, y; (1-\delta)t)\big)^{\delta/2} \\ &\leq e^{-\delta^2 t \lambda(K,D)} \sup_{x,y\in D} \big(\pi_{K,D}(x, y; (1-\delta)t)\big)^\delta \\ &\leq e^{-\delta^2 t \lambda(K,D)} \sup_{x,y\in D} \big(\pi_D(x, y; (1-\delta)t)\big)^\delta. \end{aligned} \tag{10}$$

By Eq. 3,

$$\begin{aligned} \big(\pi_D(x, y; (1-\delta)t)\big)^\delta &\leq \left(\frac{1}{|D|} + c_1 e^{-c_2(1-\delta)t}\right)^\delta \\ &\leq \frac{1}{|D|^\delta} + c_1^\delta e^{-c_2\delta(1-\delta)t},\ t \geq \frac{t_1}{1-\delta}. \end{aligned}$$

This, together with Eq. 10, gives

$$\left(\pi_{K,D}(x,y;t)\right)^{\delta} \le e^{-\delta^2 t\lambda(K,D)}\left(\frac{1}{|D|^{\delta}} + c_1^{\delta} e^{-c_2\delta(1-\delta)t}\right),\ t \ge \frac{t_1}{1-\delta}. \tag{11}$$

We obtain by Eq. 11, and by Hölder's inequality,

$$\begin{aligned}
&\int_{t_1/(1-\delta)}^{\infty} dt \int_{D\setminus K} dy\, \pi_{K,D}(x,y;t)\\
&\le \int_{t_1/(1-\delta)}^{\infty} dt \int_{D\setminus K} dy\, \left(\pi_{K,D}(x,y;t)\right)^{1-\delta} e^{-\delta^2 t\lambda(K,D)}\left(\frac{1}{|D|^{\delta}} + c_1^{\delta} e^{-c_2\delta(1-\delta)t}\right)\\
&\le \int_{t_1/(1-\delta)}^{\infty} dt \int_{D} dy\, \left(\pi_{D}(x,y;t)\right)^{1-\delta} e^{-\delta^2 t\lambda(K,D)}\left(\frac{1}{|D|^{\delta}} + c_1^{\delta} e^{-c_2\delta(1-\delta)t}\right)\\
&\le \int_{t_1/(1-\delta)}^{\infty} dt \left(\int_{D} dy\, \pi_{D}(x,y;t)\right)^{1-\delta} |D|^{\delta} e^{-\delta^2 t\lambda(K,D)}\left(\frac{1}{|D|^{\delta}} + c_1^{\delta} e^{-c_2\delta(1-\delta)t}\right)\\
&= \frac{1}{\delta^2\lambda(K,D)} e^{-\delta^2 t_1\lambda(K,D)/(1-\delta)}\\
&\quad + c_1^{\delta}|D|^{\delta}\left(c_2\delta(1-\delta) + \delta^2\lambda(K,D)\right)^{-1} e^{-t_1(\delta c_2 + \delta^2\lambda(K,D)/(1-\delta))}\\
&\le \frac{1}{\delta^2\lambda(K,D)} + \frac{c_1^{\delta}|D|^{\delta}}{c_2\delta(1-\delta)}.
\end{aligned} \tag{12}$$

By Eqs. 7 and 12,

$$u_{K,D}(x)\lambda(K,D) \le \delta^{-2} + \left(\frac{t_1}{1-\delta} + \frac{c_1^{\delta}|D|^{\delta}}{c_2\delta(1-\delta)}\right)\lambda(K,D).$$

By taking the supremum over all $x \in D \setminus K$ we obtain

$$\|u_{K,D}\|_{\infty}\lambda(K,D) \le \delta^{-2} + \left(\frac{t_1}{1-\delta} + \frac{c_1^{\delta}|D|^{\delta}}{c_2\delta(1-\delta)}\right)\lambda(K,D).$$

Hence for $\delta \in (0,1)$ and $\varepsilon \in (0,1)$,

$$\|u_{\varepsilon K,D}\|_{\infty}\lambda(\varepsilon K,D) \le \delta^{-2} + \left(\tfrac{t_1}{1-\delta} + \tfrac{c_1^{\delta}|D|^{\delta}}{c_2\delta(1-\delta)}\right)\lambda(\varepsilon K,D). \tag{13}$$

In the lemma below we obtain an upper bound for the rate at which $\lambda(\varepsilon K, D) \downarrow 0$ as $\varepsilon \downarrow 0$.

Lemma 3 *If D is open, non-empty in $\mathbb{R}^m$, $m \ge 3$, with $|D| < \infty$, and if $K \subset D$ with* cap $(K) > 0$ *then*

$$\limsup_{\varepsilon\downarrow 0} \varepsilon^{2-m}\lambda(\varepsilon K, D) \le \frac{\text{cap}\,(K)}{|D|}. \tag{14}$$

If D is open, non-empty in $\mathbb{R}^2$, with $|D| < \infty$, and if $K \subset D$ has strictly positive logarithmic capacity, then

$$\limsup_{\varepsilon\downarrow 0} \left(\log \varepsilon^{-1}\right)\lambda(\varepsilon K, D) \le \frac{2\pi}{|D|}. \tag{15}$$

We note that (i) the constants in the right-hand sides of Eqs. 14 and 15 are well-known and sharp (see for example [7]), (ii) both formulae hold for arbitrary open and connected sets D with $|D| < \infty$, and without any regularity assumptions on ∂D. We now choose

$$\delta = 1 - |D|^{1/m}\lambda(\varepsilon K, D)^{1/2}. \tag{16}$$

Then $\delta \in (0, 1)$ for all ε sufficiently small. By Eqs. 13 and 16,

$$\|u_{\varepsilon K,D}\|_\infty \lambda(\varepsilon K, D) \le 1 + O\big(\lambda(\varepsilon K, D)^{1/2}\big). \tag{17}$$

The proof of Eq. 5 is similar to the one of Theorem 5 in [3], and Theorem 1, (0.5) in [1]. Let ψ denote the normalised first eigenfunction (positive) of the Laplacian with Neumann and Dirichlet boundary conditions on ∂D and ∂K respectively, suppressing both K and D dependence. We have by Cauchy-Schwarz's inequality that $\int_{D\setminus K} \psi \le |D \setminus K|^{1/2}$. Using

$$\psi \frac{\partial u_{K,D}}{\partial \nu} = u_{K,D} \frac{\partial \psi}{\partial \nu} = 0 \text{ on } \partial D \cup \partial K,$$

we obtain by Green's formula,

$$\begin{aligned} \lambda(K, D)\|u_{K,D}\|_\infty \int_{D\setminus K} \psi \ &\ge \ \lambda(K, D) \int_{D\setminus K} u_{K,D}\psi = -\int_{D\setminus K} u_{K,D}\Delta\psi \\ &= -\int_{D\setminus K} \psi\, \Delta u_{K,D} = \int_{D\setminus K} \psi. \end{aligned}$$

This implies the assertion.

Finally Eq. 4 follows by Eqs. 5, 17, and Lemma 3. □

3 Proof of Lemma 3 and Proposition 2

Proof of Lemma 3 Recall that $0 \in D$, and so

$$R = \min\{|y| : y \in \partial D\} > 0.$$

Since K is compact,

$$R_K = \max\{|x| : x \in K\} < \infty.$$

Let

$$\varepsilon_1 = \min\left\{1, \frac{R}{R_K}\right\}.$$

If $\varepsilon \le \varepsilon_1$ then $\varepsilon K \subset B(0; R)$. See [9] for estimates related to the proof of Lemma 3. First we consider the case $m \ge 3$. Let μ_K denote the equilibrium measure of K in $\mathbb{R}^m$, and let

$$\phi_K(x) = \frac{\Gamma((m-2)/2)}{4\pi^{m/2}} \int_K \mu_K(dy)\, |x-y|^{2-m}.$$

Then $\phi_K(x) = 1,\ x \in K, 0 < \phi_K < 1,\ x \in \mathbb{R}^m \setminus K$, and ϕ_K is smooth on the complement of K. We use $1 - \phi_K$ as a trial function in the Rayleigh-Ritz characterisation of $\lambda(K, D)$. This gives

$$\begin{aligned} \lambda(K, D) \ &= \inf_{u\in H^1(D),\, u|_K=0} \frac{\int_{D\setminus K} |\nabla u|^2}{\int_{D\setminus K} u^2} \\ &\le \frac{\int_{D\setminus K} |\nabla \phi_K|^2}{\int_{D\setminus K}(1-\phi_K)^2} \\ &\le \frac{\int_{\mathbb{R}^m\setminus K} |\nabla \phi_K|^2}{\int_{D\setminus K}(1-\phi_K)^2} \\ &= \frac{\text{cap}\,(K)}{\int_{D\setminus K}(1-\phi_K)^2}. \end{aligned} \tag{18}$$

It remains to bound the denominator in the right-hand side of Eq. 18 from below. Since we will apply this lower bound with $\varepsilon_1 K$ rather than K itself, we assume that $K \subset B(0; R)$. We let $0 < \alpha < 1$. It is a standard fact that the capacitary potential is monotone increasing in K. In particular,

$$\phi_K(x) \le \phi_{B(0;R)}(x) = \min\left\{1, \left(\frac{R}{|x|}\right)^{m-2}\right\}.$$

Hence

$$\begin{aligned}\int_{D\setminus K}(1-\phi_K)^2 &\ge (1-\alpha)^2\int_{\{\phi_K(x)\le\alpha\}\cap D} 1\\ &\ge (1-\alpha)^2\big(|D| - |\{\phi_{B(0;R)}(x) > \alpha\}|\big)\\ &\ge (1-\alpha)^2\big(|D| - \alpha^{-m/(m-2)}\omega_m R^m\big),\end{aligned} \tag{19}$$

where $\omega_m = |B_1(0)|$. We choose α such that

$$\alpha = \alpha^{-m/(m-2)}\frac{|B(0;R)|}{|D|}. \tag{20}$$

This, together with Eqs. 18, 19 and 20 implies

$$\lambda(K, D) \le \frac{\mathrm{cap}\,(K)}{|D|}\left(1 - \left(\frac{|B(0;R)|}{|D|}\right)^{(m-2)/(2(m-1))}\right)^{-3}. \tag{21}$$

In particular for $\varepsilon \in (0, 1]$, $\varepsilon\varepsilon_1 K \subseteq \varepsilon B(0; R)$, and this together with Eq. 21 gives

$$\lambda(\varepsilon\varepsilon_1 K, D) \le \frac{\mathrm{cap}\,(\varepsilon\varepsilon_1 K)}{|D|}\left(1 - \left(\frac{\varepsilon|B(0;R)|}{|D|}\right)^{(m-2)/(2(m-1))}\right)^{-3}. \tag{22}$$

Formula Eq. 14 follows by Eq. 22, and scaling of the Newtonian capacity,

$$\mathrm{cap}\,(\varepsilon K) = \varepsilon^{m-2}\mathrm{cap}\,(K).$$

Next we consider the planar case $m = 2$. We use Hadamard's method of descent so as to avoid logarithmic potential theory. See for example p.51 in [9]. Let $h \ge R$, and consider the cylinder $(D \setminus K) \times (0, h) \subset \mathbb{R}^3$. Then the first eigenvalue of the Laplacian acting in $L^2(D\setminus K))$ with Dirichlet boundary condition on ∂K, and Neumann boundary condition on ∂D is precisely equal to the first eigenvalue of the Laplacian acting in $L^2((D \setminus K) \times (0, h))$ with Dirichlet boundary condition on $\partial(K \times (0, h))$, and Neumann boundary condition on $\partial(D \times (0, h)) \setminus \partial(K \times (0, h))$. We apply Eq. 21 to the setting above and obtain by monotonicity of Newtonian capacity,

$$\begin{aligned}\lambda(\varepsilon\varepsilon_1 K, D) &\le \lambda(\varepsilon B(0; R), D)\\ &\le \frac{\mathrm{cap}\,(B(0;\varepsilon R)\times(0,h))}{|D|h}\left(1 - \left(\frac{\varepsilon|B(0;R)|}{|D|}\right)^{1/4}\right)^{-3}.\end{aligned} \tag{23}$$

To obtain an upper bound on $\mathrm{cap}\,(B(0; \varepsilon R) \times (0, h))$ we let $C(R', h') \subset \mathbb{R}^3$ be an ellipsoid with a circular cross section of radius R' and axis h'. Then for a suitable translation and rotation $C(R', h') \supset B(0; \varepsilon R) \times (0, h)$ provided

$$\frac{h^2}{h'^2} + \frac{(\varepsilon R)^2}{R'^2} \le 1. \tag{24}$$

We let $\alpha \in (0, 1)$ be arbitrary, and choose

$$R' = \varepsilon^{-\alpha}(\varepsilon R), \tag{25}$$

and

$$h' = \left(1 - \varepsilon^{2\alpha}\right)^{-1/2} h. \tag{26}$$

The choice Eqs. 25–26 satisfies Eq. 24. For $\frac{h'}{R'} \to \infty$, or equivalently $\varepsilon \downarrow 0$ with h fixed, we have by formula (12) on p.260 in [6],

$$\begin{aligned} \operatorname{cap}\left(C(R', h')\right) &= \frac{2\pi h'}{\log(h'/R')}(1 + o(1)) \\ &\le \frac{2\pi h}{\left(1 - \varepsilon^{2\alpha}\right)^{1/2} \log(h/R')}(1 + o(1)) \\ &\le \frac{2\pi h}{(1-\alpha)\left(1 - \varepsilon^{2\alpha}\right)^{1/2} \log \varepsilon^{-1}}(1 + o(1)). \end{aligned}$$

Thus,

$$\frac{\operatorname{cap}\left(B(0; \varepsilon R) \times (0, h)\right)}{|D| h} \le \frac{2\pi}{(1-\alpha)|D| \log \varepsilon^{-1}}(1 + o(1)).$$

By Eq. 23,

$$\limsup_{\varepsilon \downarrow 0} \left(\log \varepsilon^{-1}\right) \lambda(\varepsilon \varepsilon_1 K, D) \le \frac{2\pi}{(1-\alpha)\,|D|}.$$

Since $\alpha \in (0, 1)$ was arbitrary, this completes the proof of the case $m = 2$. □

Proof of Proposition 2 To prove (i) we recall that, since D is open, D is a countable union of open components. Suppose that this union contains at least two elements, one of which is C. Then both C and $D \setminus C$ are open and non-empty. Let 1_A denote the indicator function of a set A. From Eq. 3 we obtain,

$$\left| \int_C dy\, \pi_D(x, y; t) - \frac{|C|}{|D|} \right| \le c_1 |C| e^{-c_2 t},\ t \ge t_1,\ x \in D.$$

We note that

$$q_{C,D}(x; t) = \int_C dy\, \pi_D(x, y; t)$$

is the solution of the heat equation

$$\Delta q = \frac{\partial q}{\partial t},$$

with initial condition

$$q(x; 0) = 1_C(x),$$

and with a Neumann (insulating) boundary condition on ∂D. It follows that

$$q_{C,D}(x; t) = 1_C(x),\ t > 0.$$

From Eq. 3 we have

$$\left| 1 - \frac{|C|}{|D|} \right| \le c_1 |C| e^{-c_2 t},\ t \ge t_1,\ x \in C.$$

We conclude that, by taking the limit $t \to \infty$, $|C| = |D|$. Since $C \subset D$, $|D \setminus C| = 0$. This contradicts $D \setminus C$ is open and non-empty. This in turn implies that D consists of just one component C. Hence C is connected. This implies assertion (ii). To prove (iii) we have that Eq. 3 implies

$$\int_D dx\, \pi_D(x, x; t) \le 1 + c_1 |D| e^{-c_2 t},\ t \ge t_1.$$

Hence the Neumann heat semigroup is trace-class, and

$$1 + e^{-t\mu(D)} \leq \int_D dx\, \pi_D(x, x; t) \leq 1 + c_1 |D| e^{-c_2 t},\ t \geq t_1. \tag{27}$$

Taking the limit $t \to \infty$ in Eq. 27 implies the second inequality in Eq. 6. The first inequality in Eq. 6 is due to Weinberger [11]. □

Acknowledgments MvdB acknowledges support by the Leverhulme Trust through Emeritus Fellowship EM-2018-011-9.

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